

On predictability of ultra short AR(1) sequences *

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Abstract

This paper addresses short term forecast of ultra short AR(1) sequences (4 to 6 terms only) with a single structural break at an unknown time and of unknown sign and magnitude. As prediction of autoregressive processes requires estimated coefficients, the efficiency of which relies on the large sample properties of the estimator, it is a common perception that prediction is practically impossible for such short series with structural break. However, we obtain a heuristic result that some universal predictors represented in the frequency domain allow certain predictability based on these ultra short sequences. The predictors that we use are universal in a sense that they are not oriented on particular types of autoregressions and do not require explicit modelling of structural break. The shorter the sequence, the better the one-step-ahead forecast performance of the smoothed predicting kernel. If the structural break entails a model parameter switch from negative to positive value, the forecast performance of the smoothed predicting kernel is better than that of the linear predictor that utilize AR(1) coefficient estimated from the ultra short sequence without taking the structural break into account regardless whether the innovation terms in the learning sequences are constructed from independent and identically distributed random Gaussian or Gamma variables, scaled pseudo-uniform variables, or first-order auto-correlated Gaussian process.

Keywords: predicting, non-stationarity, structural break, autoregressive process.

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1 Introduction

In this paper, we readdress the problem of one-step-ahead forecast of a first order autoregressive process, $AR(1)$, with one structural break, i.e., a permanent change, in the $AR(1)$ model parameter. Specifically, we consider the scenario where the learning sequence, i.e., the segment of time series process used for model estimation and forecast, is very short, and where the structural break occurs at a random time point in the learning sequence.

Forecasting autoregressive process is a well developed area with well known results. One strand of literature addresses this problem via the time series models that are primarily specified from the time-domain modelling perspective (see, among many others, Box and Jenkins, 1976; Abraham and Ledolter, 1986; Stine, 1987; Cryer et al., 1990; Cortez et al., 2004; Hamilton, 1994; Xia and Zheng, 2015, and the references therein), or via the exponential smoothing and the filtering techniques constructed based on state-space approach where the smoothers and filters are primarily characterized in the time-domain (see, e.g., Roberts, 1982; Williams, 1987; Paige and Saunders, 1977; Chatfield and Yar, 1988; Ord et al., 1997; Chatfield et al., 2001; Hyndman et al., 2002; Bermúdez et al., 2006; Hyndman et al., 2008, and references therein). A separate yet related strand of literature addresses this problem via smoothing and filtering techniques where the smoothers and filters are primarily characterized in the frequency-domain (see, e.g., Cambanis and Soltani, 1984; Ledolter and Kahl, 1984; Lyman and Edmonson, 2001; Dokuchaev, 2012; 2014; 2016, and references therein). In this paper, we address this problem via the convolution of a near-ideal causal smoothing filter and a predicting kernel that are primarily characterized in the frequency-domain (Dokuchaev, 2012; 2014; 2016).

Many strategies have been proposed to address the practical concern of possible model parameters structural break in the learning sequence that may compromise modelling efficiency and forecast performance of the time series model (see, among many others, Bagshaw and Johnson, 1977; Sastri, 1986; Andrews, 1993; Bai and Perron, 1998; 2003; Pesaran and Timmermann, 2004; Clements and Hendry, 2006; Davis et al., 2006; Lin and Wei, 2006; Kim et al., 2009; Rossi, 2013;

Pesaran et al., 2013, and the references therein). Implementation of these strategies require the availability of learning sequences that are considerably longer than those considered in this paper. We cite a few examples. The method proposed by Bai and Perron (1998; 2003) to estimate the timing of structural break requires at least 10 observations on either side of the break. Pesaran and Timmermann (2007) simulated random processes that each contain 100 to 200 observations to mimic learning sequences with structural break in AR(1) model parameter in order to assess the performance of their proposed set of cross-validation and forecast combination procedures that use pre-break and post-break data to perform time series forecast. Giraitis et al. (2013) simulated time series processes that each contain 200 observations to mimic learning sequences with structural break in the mean of the simulated random processes in order to assess the performance of their proposed one-step-ahead forecast algorithms based on adaptive linear filtering.

In this paper, we consider the scenario when the learning sequence only contain 4 to 6 data points, and, as such, are too short to effectively apply structural break timing estimation strategies, and to efficiently estimate pre-break and post-break AR(1) model parameters. We consider a family of linear filters proposed by Dokuchaev (2016) where the impulse response function is obtained by inverse Z-transform of the product between the transfer function of a family of near-ideal causal smoothers (Dokuchaev, 2016) and the transfer function of a family of predicting kernels (Dokuchaev, 2014). The Monte Carlo experiments reported in Dokuchaev (2016) have demonstrated clear advantage of using the convolution of the near-ideal causal smoother and the predicting kernel compared to using the predicting kernel alone to generate the impulse response function for the linear predictor in terms of one-step-ahead forecast performance of AR(2) processes without structural break. However, their relative forecast performance have never been assessed in the context of AR(1) processes with a single, unknown, random time point structural break in a very short learning sequence. Additionally, numerical experiments reported in Dokuchaev (2016) utilize 100 observations in the learning sequence. This begs the question whether the proposed linear predictor will perform well in the context of a very short learning sequence with structural break. This paper seeks to close this gap in the literature.

Following the choice of benchmark used in, among others, Pesaran and Timmermann (2007) and

Giraitis et al. (2013), we use the one-step-ahead forecasts from an AR(1) model that ignores structural break and utilize all observations, pre-break and post-break, as our benchmark since this is an appropriate model to use in situations with no breaks. The main contribution of this paper is our demonstration via simulation experiment that the one-step-ahead forecast performance of this family of linear filters is better than that of our chosen benchmark. Additionally, its performance is comparable to that of the one-step-ahead forecasts from an AR(1) model with the same model parameter as the synthetic AR(1) model parameter used to simulate the post-break random process.

The rest of the paper is as follows. Section 2 details the problem formulation. Section 3 presents the Monte Carlo simulation results. Section 4 concludes.

2 Problem setting

Consider a stochastic discrete time process described by AR(1) autoregression

$$\begin{aligned} x(t) &= \beta(t) x(t-1) + \sigma \eta(t), \quad t = 0, \dots, d-1 \quad x(-1) = 0, \\ \beta(t) &= \begin{cases} \beta_1, & t < \theta, \\ \beta_2, & t \geq \theta. \end{cases} \end{aligned} \quad (1)$$

where $\beta_1 \in (\beta_{\min}, \beta_{\max})$, $\beta_2 \in (\beta_{\min}, \beta_{\max})$, $\beta_{\min} < \beta_{\max}$, $|\beta_{\min}| < 1$, $|\beta_{\max}| < 1$, $\sigma \in (0, \infty)$, and $\eta(t)$ is the innovation term of the time series. This model features a single random structural break to take place at a random time θ with the values in the set $\{1, \dots, d-2\}$. We assume that $\eta(t)$ are mutually independent for all t and independent on η .

We consider predicting problem for this process in the case where an ultra short sequences with no more than six data points are available.

2.1 Special predictors

We investigate the performance of linear time-invariant predictors with an output

$$y(t) = \sum_{\tau=0}^t h(t-\tau) x(\tau), \quad t \leq d-1, \quad (2)$$

where $d \leq 6$. The process $y(t)$ is supposed to approximate the process $x(t + 1)$, i.e., (2) represents a one-step-ahead predictor. The predictor is defined by a impulse response function $h : \mathbb{Z} \rightarrow \mathbb{R}$, where \mathbb{Z} is the set of integers, and \mathbb{R} is the set of real numbers.

In our experiments, we calculate predicting kernels h via their Z-transforms that are represented explicitly, such that

$$h(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{i\omega}) e^{i\omega t} d\omega, \quad t \in \mathbb{Z}. \quad (3)$$

Here complex-valued functions $H : \mathbb{C} \rightarrow \mathbb{C}$ are transfer functions of the corresponding predictors.

In our experiments, we used two different transfer functions

$$H(z) = K(z), \quad (4)$$

and

$$H(z) = K(z) F(z). \quad (5)$$

Here $z \in \mathbb{C}$,

$$K(z) = z \left(1 - \exp \left[-\frac{\gamma}{z + 1 - \gamma^{-r}} \right] \right). \quad (6)$$

The function $K(z)$ is the transfer function of an one-step predictor from Dokuchaev (2016); $r > 0$, $\gamma > 0$ are the parameters.

In (5),

$$\begin{aligned} F(z) &= \left(\exp \frac{(1-a)^p}{z+a} + G(z) \right)^m, \\ G(z) &= -\xi(a, p) + \frac{\gamma(a, p)}{N} \left((-1)^N z^{-N} - 1 \right), \\ \xi(a, p) &= \exp[-(1-a)^{p-1}], \\ \gamma(a, p) &= |1-a|^{p-2} \xi(a, p). \end{aligned} \quad (7)$$

Here $a \in (0, 1)$, $p \in (1/2, 1)$, $m \geq 1$, and $N \geq 1$, are the parameters, $m, n \in \mathbb{Z}$. The function $F(z)$ is the transfer function for a smoothing filter introduced in Dokuchaev (2016).

It can be noted that these linear predictors were constructed for semi-infinite one-sided sequences, since the corresponding kernels $h(t)$ have infinite support on \mathbb{Z} . In theory, the performance of these

predictors is robust with respect to truncation; see the discussion on robustness in Dokuchaev (2012) and Dokuchaev (2016). However, we found, as a heuristic result of this paper, that their application to the ultra short series also brings some positive result, meaning that these sequences feature some predictability. Worthy of note again is that implementation of these predictors does not involve explicit modelling of and adjustment for structural break, including break time and magnitude. Moreover, these predictors do not even require that the underlying process is an autoregressive process or any other particular kind of a processes.

2.2 Comparison with other predictors

We compare the performance of our predictors with an “ideal” linear predictor

$$y_{ideal}(d-1) = \beta(d) x(d-1) , \quad (8)$$

where $\beta(d) = \beta_2$ is the post-break AR(1) model parameters that generate the post-break observations $x(d-1)$ and $x(d)$. This predictor is not feasible unless $\beta(d)$ is supposed to be known. In our setting, $\beta(d)$ is unknown and has to be estimated from the observations. We will use the performance of this predictor as a benchmark.

Additionally, we will compare the performance of our predictors with the performance of the predictor

$$y_{AR(1)}(d-1) = \hat{\beta}(d) x(d-1) , \quad (9)$$

where $\hat{\beta}(d)$ is estimated by fitting an AR(1) model to the sequence $\{x(\tau)\}_{\tau=0}^{d-1}$ that involve pre-break and post-break observations using the build-in function `ar.ols()` in the R computing environment (R Core Team, 2016) implementing the ordinary least squares model parameter estimation strategy (pp. 368-370, Luetkepohl, 2008). This is an appropriate model estimation procedure to use if the sequence $\{x(\tau)\}_{\tau=0}^{d-1}$ does not contain structural break. By choosing this predictor as the benchmark for our numerical experiment, we seek to address the question of how costly is it to ignore breaks when performing one-step-ahead forecasting the direction of a time series using the prediction algorithms considered, i.e., (4), (5), and (8), relative to (9).

3 Simulation experiment

We perform simulation experiments to investigate the one-step ahead forecast performance of (4), (5), and (8), relative to (9) in predicting $x(d)$, given $\{x(\tau)\}_{\tau=0}^{d-1}$ simulated from (1) using four different specifications of (β_1, β_2)

1. $\beta_1 \in (0, 1), \beta_2 \in (0, 1)$,
2. $\beta_1 \in (-1, 1), \beta_2 \in (-1, 1)$,
3. $\beta_1 \in (-1, 0), \beta_2 \in (0, 1)$,
4. $\beta_1 \in (0, 1), \beta_2 \in (-1, 0)$,

and four different specifications of $\eta(t)$

1. Independent and identically distributed (IID) Gaussian innovations: In this setting, we specify

$$\eta(t) \sim \mathcal{N}(0, 1) \quad (10)$$

as IID random numbers drawn from the standard Gaussian distribution.

2. IID shifted Gamma innovation: In this setting, we specify

$$\eta(t) = \gamma_0(t) - \sqrt{2} \quad (11)$$

where $\{\gamma_0(t)\}_{t=0}^{d-1}$ were random numbers drawn randomly from Gamma distribution with shape parameter 2 and scale parameter $2^{-1/2}$, i.e., $\Gamma(2, 2^{-1/2})$.

3. IID scaled pseudo-uniform innovation: In this setting, we specify

$$\eta(t) = \sqrt{12} (\exp(t + 3 \arctan(s)) - \lfloor \exp(t + 3 \arctan(s)) \rfloor - 1/2) \quad (12)$$

where $t = 1, \dots, d-1$, $s = 1, \dots, N_{sim}$, and where N_{sim} is the total number of simulations to be performed.

4. Auto-correlated Gaussian innovation: In this setting, we specify

$$\eta(t) = 2^{-1/2} (\eta_0(t) + \eta_0(t-1)) \quad (13)$$

where $\{\eta_0(t)\}_{t=0}^{d-1}$ were IID random numbers drawn randomly from $\mathcal{N}(0, 1)$ and the lag-one auto-correlation is $E[\eta(t)\eta(t-1)] = 0.5$.

For this simulation experiment, the linear predictors (4) and (5) are implemented in the form of (2) as

$$y(d-1) = \sum_{\tau=0}^{d-1} h(t-\tau) x(\tau) \approx x(d) , \quad (14)$$

where $y(d-1) = y_{KH}(d-1)$ and $y(d-1) = y_K(d-1)$ are the one-step-ahead forecasts, and where $h(t-\tau) = h_{KH}(t-\tau)$ and $h(t-\tau) = h_K(t-\tau)$ are the impulse response functions for (4) and (5) respectively. Following the choice of parameters used in Dokuchaev (2016), we set $a = 0.6, p = 0.7, N = 100, m = 2$, for the smoothing filter (7), and set $\gamma = 1.1$, for the predicting kernel (6). We investigate the sensitivity of the predicting kernel for some different values of r , where $r \in \{0.8, 1.1, 1.5, 2\}$, and we consider three different lengths of ultra short sequence d , where $d \in \{4, 5, 6\}$.

The ideal linear predictor (8) is implemented as

$$y_{ideal}(d-1) = \beta_2(d) x(d-1) \approx x(d) . \quad (15)$$

For this predictor, one needs to know $\beta_2(d)$, i.e., the post-break AR(1) model parameters used to simulate $x(d)$. In practice, it is impossible to know $\beta_2(d)$. We include (15) as it represents a theoretical ideal benchmark.

The linear predictor (9) is implemented as

$$y_{AR(1)}(d) = \hat{\beta}(d) x(d-1) \approx x(d) , \quad (16)$$

where $\hat{\beta}(d)$ is estimated by fitting an AR(1) model to the learning sequence $\{x(\tau)\}_{\tau=0}^{d-1}$. This is a commonly used approach in AR(1) time series forecasting, and one that depends on the large-sample properties of the available time series for efficient model parameter estimation. We are interested

to investigate the finite-sample properties in terms of forecast performance of (14) relative to (16) in the context of ultra short learning sequences considered in this paper.

For each combination of (β_1, β_2) , $\eta(t)$, r , and d , we perform N_{sim} simulations where $N_{sim} \in \{1 \times 10^5, 2 \times 10^5, 3 \times 10^5\}$. For each simulation, we simulate an AR(1) processes with a single random structural break at random unknown time point following the data generation process (1), each containing $d + 1$ observations. where $\beta_1(t)$, $\beta_2(t)$, σ , and $\eta(t)$ are mutually independent. The first d observations are used as the sequence based on which we forecast the $(d + 1)$ -th observation.

Let $\mathbb{E}[\cdot]$ denote the sample mean across the N_{sim} Monte Carlo trials performed for each scenario indexed by s where $s = 1, \dots, N_{sim}$. Specifically, we let

$$\begin{aligned} e_{KH} &= \left(\mathbb{E} (x(d) - y_{KH}(d-1))^2 \right)^{1/2}, \\ e_K &= \left(\mathbb{E} (x(d) - y_K(d-1))^2 \right)^{1/2}, \\ e_{ideal} &= \left(\mathbb{E} (x(d) - y_{ideal}(d-1))^2 \right)^{1/2}, \\ e_{AR(1)} &= \left(\mathbb{E} (x(d) - y_{AR(1)}(d-1))^2 \right)^{1/2}, \end{aligned}$$

be the sample root-mean-squared error (RMSE) for (14) that implement (4) and (5), (15), and (16) respectively.

We carry out the simulation experiments in the R computing environment (R Core Team, 2016). Simulation of the learning sequence is carried out by iterative application of (1). The estimation of AR(1) parameter $\hat{\beta}$ for the implementation of (16) is performed using the `ar.ols()` script in R. Numerical integrations carried out to map (4) and (5) to their respective impulse response functions to be used in (14) are implemented via the `myintegrate()` script in the R add-on package `elliptic` proposed in Hankin (2006).

Table 1 depicts the simulation experiments results for the setting with $\beta_1, \beta_2 \in (0, 1)$ and $\eta(t) \sim \mathcal{N}(0, 1)$. For a short sequence of length $d = 4, 5, 6$, and for the four different values of predicting kernel parameter r , $r = 0.8, 1.1, 1.5, 2$, the RMSE of the smoothed predicting kernel linear predictor, is smaller than the RMSE of the linear predictor that utilize AR(1) model parameter estimated based on the learning sequence ignoring the presence of structural break (16).

The shorter the learning sequence, the better the performance of the smoothed predicting kernel linear predictor. This trend is consistent across three different sizes of Monte Carlo simulation $N_{sim} \in \{1 \times 10^4, 2 \times 10^4, 3 \times 10^4\}$.

Worthy of note is that this smoothed predictor does not require explicit modelling of structural break. In practice, when the available learning sequence is short, and the model parameter structural break time and magnitude uncertain, it is not possible to efficiently apply structural break estimation and adjustment procedures for parameter estimation and time series forecasting due to series length constraint. In this context, the smoothed predicting kernel (5) appears to be an alternative approach that may offer satisfactory forecast performance, circumventing the need of resorting to model parameter estimation that ignore structure break.

The fact that the RMSE of the predicting kernel linear predictor without smoothing (4) is larger than the RMSE of the linear predictor (16) highlights the role of the near-ideal causal smoother (7) in improving the forecast performance of (4). By dampening the high frequency noise, the smoothed prediction kernel is more able to capture the salient features of the simulated AR(1) process with random structural break based on a short learning sequence in order to deliver better one-step-ahead forecast performance than the linear predictor (16). Without the aid of the smoothing kernel, the performance of the predicting kernel (4) is, in general, even poorer than that of the linear predictor (16) that relies on model parameter estimate from an AR(1) model that ignores structural break.

It is not surprising that the performance of the linear predictor (16) is poorer than the ideal predictor (15). Utilizing pre-break and post-break data to estimate post-break model parameter when the break time and magnitude are unknown inevitably leads to parameter estimation error. Although cross-validation methods have been proposed to utilize pre-break and post-break data to use pre-break data to estimate the parameters of the model used to compute out-of sample forecasts (see, e.g., Pesaran and Timmermann, 2007), the number of observations required to implement these methodologies is considerably larger than those we consider in this paper.

Table 2 depicts the results of simulations with $\eta(t) \sim \mathcal{N}(0, 1)$ for three other model parameter settings, i.e., a wider range of possible model parameters with $\beta_1, \beta_2 \in (-1, 1)$, a model parameter shift from negative to positive values $\beta_1 \in (-1, 0), \beta_2 \in (0, 1)$, and a model parameter shift from positive to negative values $\beta_1 \in (0, 1), \beta_2 \in (-1, 0)$. The forecast performance of the smoothed predicting kernel linear predictor (5) appears to be dependent on the sign of β_2 . If $\beta_2 \in (0, 1)$, (5) performs better than (16), and vice versa.

Table 3 depicts a subset of the simulation results pertaining to the setting where $\eta(t)$ is as defined in (11), while Table 4 depicts those pertaining to the setting where $\eta(t)$ is as defined in (13). They shows similar trends as those demonstrated for those of (10) as depicted in Table 1 and Table 2 above.

However, the numerical results pertaining to simulation scenarios with IID innovation terms are in some ways different from those with correlated innovation terms. Table 5 depicts a subset of the simulation results pertaining to the setting where $\eta(t)$ is as defined in (12) where $E[\eta(t)\eta(t-1)] = 0.5$. While the forecast performance of the smoothed predicting kernel (5) is still better than the linear predictor (16) for $\beta_1 \in (-1, 0)$ and $\beta_2 \in (0, 1)$ in this context where the innovation terms are auto-correlated, it is not so for the remaining three simulation scenarios.

Table 1: One-step-ahead forecast performance with $\beta_1, \beta_2 \in (0, 1)$, random structural break at $\theta \in \{2, \dots, d-2\}$, and $\eta(t) \sim \mathcal{N}(0, 1)$.

	e_{ideal}	$e_{AR(1)}$	e_K	e_{KH}	$e_{ideal}/e_{AR(1)}$	$e_K/e_{AR(1)}$	$e_{KH}/e_{AR(1)}$
Panel (a): $\beta_1, \beta_2 \in (0, 1), N_{sim} = 1 \times 10^5$							
$r = 0.8, d = 4$	0.30031	0.44385	0.39284	0.34568	0.67661	0.88506	0.77882
$r = 0.8, d = 5$	0.29881	0.37738	0.39343	0.34661	0.79182	1.04254	0.91847
$r = 0.8, d = 6$	0.29970	0.35256	0.39499	0.34795	0.85004	1.12032	0.98691
$r = 1.1, d = 4$	0.30082	0.44232	0.39817	0.34568	0.68010	0.90018	0.78152
$r = 1.1, d = 5$	0.29971	0.37597	0.39765	0.34586	0.79717	1.05767	0.91991
$r = 1.1, d = 6$	0.29986	0.35161	0.39970	0.34686	0.85283	1.13678	0.98640
$r = 1.5, d = 4$	0.30066	0.46608	0.40348	0.34393	0.64508	0.86568	0.73791
$r = 1.5, d = 5$	0.29979	0.37684	0.40480	0.34393	0.79554	1.07420	0.91267
$r = 1.5, d = 6$	0.29939	0.35283	0.40506	0.34468	0.84853	1.14802	0.97689
$r = 2, d = 4$	0.30010	0.44740	0.41242	0.34097	0.67077	0.92182	0.76210
$r = 2, d = 5$	0.30127	0.37954	0.41351	0.34376	0.79379	1.08950	0.90572
$r = 2, d = 6$	0.30108	0.35406	0.41589	0.34351	0.85037	1.17464	0.97022
Panel (b): $\beta_1, \beta_2 \in (0, 1), N_{sim} = 2 \times 10^5$							
$r = 0.8, d = 4$	0.29981	0.44456	0.39188	0.34614	0.67439	0.88152	0.77862
$r = 0.8, d = 5$	0.29951	0.37979	0.39302	0.34690	0.78861	1.03484	0.91340
$r = 0.8, d = 6$	0.30024	0.35278	0.39404	0.34810	0.85107	1.11696	0.98673
$r = 1.1, d = 4$	0.29946	0.44962	0.39626	0.34470	0.66603	0.88132	0.76666
$r = 1.1, d = 5$	0.29959	0.38039	0.39761	0.34553	0.78759	1.04526	0.90836
$r = 1.1, d = 6$	0.29981	0.35372	0.39946	0.34672	0.84759	1.12934	0.98021
$r = 1.5, d = 4$	0.30063	0.44241	0.40416	0.34362	0.67954	0.91355	0.77671
$r = 1.5, d = 5$	0.30073	0.38054	0.40539	0.34494	0.79027	1.06529	0.90643
$r = 1.5, d = 6$	0.29988	0.35258	0.40506	0.34588	0.85053	1.14885	0.98100
$r = 2, d = 4$	0.30021	0.44961	0.41343	0.34147	0.66770	0.91953	0.75948
$r = 2, d = 5$	0.29953	0.37829	0.41324	0.34205	0.79179	1.09237	0.90420
$r = 2, d = 6$	0.30112	0.35523	0.41516	0.34450	0.84766	1.16869	0.96978
Panel (c): $\beta_1, \beta_2 \in (0, 1), N_{sim} = 3 \times 10^5$							
$r = 0.8, d = 4$	0.30000	0.44340	0.39318	0.34638	0.67659	0.88675	0.78119
$r = 0.8, d = 5$	0.29977	0.38060	0.39391	0.34738	0.78762	1.03496	0.91271
$r = 0.8, d = 6$	0.30012	0.35344	0.39397	0.34852	0.84913	1.11468	0.98607
$r = 1.1, d = 4$	0.30019	0.44827	0.39724	0.34511	0.66967	0.88617	0.76988
$r = 1.1, d = 5$	0.30003	0.37772	0.39832	0.34573	0.79431	1.05454	0.91530
$r = 1.1, d = 6$	0.30053	0.35391	0.39903	0.34743	0.84917	1.12750	0.98171
$r = 1.5, d = 4$	0.30025	0.44945	0.40429	0.34404	0.66804	0.89953	0.76547
$r = 1.5, d = 5$	0.29953	0.37719	0.40392	0.34392	0.79412	1.07088	0.91180
$r = 1.5, d = 6$	0.29961	0.35373	0.40467	0.34501	0.84700	1.14402	0.97537
$r = 2, d = 4$	0.30004	0.44259	0.41150	0.34183	0.67793	0.92975	0.77233
$r = 2, d = 5$	0.29985	0.37641	0.41410	0.34193	0.79660	1.10013	0.90838
$r = 2, d = 6$	0.29997	0.35304	0.41438	0.34299	0.84967	1.17374	0.97152

Table 2: Random structural break at $\theta \in \{2, \dots, d-2\}$, $\eta(t) \sim \mathcal{N}(0, 1)$ and $N_{sim} = 3 \times 10^5$.

	e_{ideal}	$e_{AR(1)}$	e_K	e_{KH}	$e_{ideal}/e_{AR(1)}$	$e_K/e_{AR(1)}$	$e_{KH}/e_{AR(1)}$
Panel (a): $\beta_1, \beta_2 \in (-1, 1)$							
$r = 0.8, d = 4$	0.29946	0.44548	0.72830	0.42322	0.67221	1.63486	0.95002
$r = 0.8, d = 5$	0.29988	0.38167	0.75529	0.42575	0.78570	1.97890	1.11548
$r = 0.8, d = 6$	0.29970	0.36081	0.77564	0.42754	0.83064	2.14972	1.18496
$r = 1.1, d = 4$	0.29967	0.44583	0.73269	0.42220	0.67215	1.64341	0.94699
$r = 1.1, d = 5$	0.30002	0.38609	0.76609	0.42452	0.77705	1.98421	1.09952
$r = 1.1, d = 6$	0.29954	0.36060	0.79259	0.42832	0.83065	2.19794	1.18778
$r = 1.5, d = 4$	0.29981	0.45026	0.75434	0.42250	0.66587	1.67533	0.93834
$r = 1.5, d = 5$	0.30058	0.38192	0.78852	0.42577	0.78702	2.06463	1.11482
$r = 1.5, d = 6$	0.29974	0.36132	0.81562	0.42766	0.82957	2.25735	1.18361
$r = 2, d = 4$	0.29950	0.45461	0.77642	0.42232	0.65880	1.70790	0.92899
$r = 2, d = 5$	0.29958	0.38185	0.81207	0.42434	0.78454	2.12665	1.11127
$r = 2, d = 6$	0.29965	0.36041	0.84057	0.42723	0.83141	2.33228	1.18541
Panel (b): $\beta_1 \in (-1, 0), \beta_2 \in (0, 1)$							
$r = 0.8, d = 4$	0.30020	0.44990	0.39408	0.34707	0.66727	0.87593	0.77143
$r = 0.8, d = 5$	0.30020	0.44990	0.39408	0.34707	0.66727	0.87593	0.77143
$r = 0.8, d = 6$	0.29935	0.36745	0.39601	0.34919	0.81467	1.07770	0.95028
$r = 1.1, d = 4$	0.29977	0.44824	0.39914	0.34585	0.66876	0.89047	0.77157
$r = 1.1, d = 5$	0.29969	0.38747	0.40141	0.34654	0.77346	1.03598	0.89435
$r = 1.1, d = 6$	0.30021	0.36857	0.40261	0.34920	0.81455	1.09238	0.94745
$r = 1.5, d = 4$	0.30021	0.44596	0.40667	0.34529	0.67318	0.91189	0.77426
$r = 1.5, d = 5$	0.29937	0.38618	0.40881	0.34472	0.77521	1.05861	0.89264
$r = 1.5, d = 6$	0.30039	0.36888	0.41055	0.34667	0.81433	1.11297	0.93979
$r = 2, d = 4$	0.30062	0.47598	0.41782	0.34428	0.63158	0.87781	0.72331
$r = 2, d = 5$	0.30000	0.38460	0.42137	0.34384	0.78005	1.09562	0.89404
$r = 2, d = 6$	0.29983	0.36798	0.42229	0.34552	0.81481	1.14760	0.93896
Panel (c): $\beta_1 \in (0, 1), \beta_2 \in (-1, 0)$							
$r = 0.8, d = 4$	0.29982	0.44890	0.94541	0.48679	0.66790	2.10604	1.08439
$r = 0.8, d = 5$	0.29966	0.38576	0.98889	0.49084	0.77681	2.56348	1.27239
$r = 0.8, d = 6$	0.30018	0.36923	1.01892	0.49363	0.81300	2.75955	1.33692
$r = 1.1, d = 4$	0.30018	0.45163	0.95963	0.48872	0.66466	2.12480	1.08212
$r = 1.1, d = 5$	0.29999	0.38558	1.00126	0.49075	0.77802	2.59674	1.27275
$r = 1.1, d = 6$	0.29929	0.36789	1.03947	0.49352	0.81353	2.82552	1.34151
$r = 1.5, d = 4$	0.30073	0.44550	0.97856	0.48733	0.67504	2.19655	1.09391
$r = 1.5, d = 5$	0.29968	0.38577	1.02990	0.49081	0.77683	2.66972	1.27229
$r = 1.5, d = 6$	0.29997	0.36922	1.07103	0.49593	0.81244	2.90078	1.34319
$r = 2, d = 4$	0.29985	0.45724	1.00339	0.48925	0.65579	2.19445	1.07001
$r = 2, d = 5$	0.29995	0.38617	1.05628	0.49223	0.77673	2.73525	1.27464
$r = 2, d = 6$	0.29945	0.36725	1.10925	0.49614	0.81538	3.02041	1.35097

Table 3: One-step-ahead forecast performance with random structural break at $\theta \in \{2, \dots, d-2\}$, and $\eta(t) = \gamma_0(t) - \sqrt{2}$, where $\gamma_0(t) \sim \Gamma(2, 2^{-1/2})$, and $N_{sim} = 3 \times 10^5$.

	e_{ideal}	$e_{AR(1)}$	e_K	e_{KH}	$e_{ideal}/e_{AR(1)}$	$e_K/e_{AR(1)}$	$e_{KH}/e_{AR(1)}$
Panel (a): $\beta_1, \beta_2 \in (0, 1)$							
$r = 0.8, d = 4$	0.30084	0.53991	0.39310	0.34721	0.55721	0.72808	0.64308
$r = 0.8, d = 5$	0.30016	0.42423	0.39442	0.34722	0.70754	0.92973	0.81846
$r = 0.8, d = 6$	0.29990	0.37654	0.39329	0.34776	0.79645	1.04447	0.92357
$r = 2, d = 4$	0.29977	0.52893	0.41234	0.34127	0.56676	0.77957	0.64520
$r = 2, d = 5$	0.30001	0.41651	0.41255	0.34243	0.72029	0.99049	0.82214
$r = 2, d = 6$	0.30028	0.37886	0.41402	0.34331	0.79259	1.09281	0.90616
Panel (b): $\beta_1, \beta_2 \in (-1, 1)$							
$r = 0.8, d = 4$	0.29997	0.56377	0.72394	0.42278	0.53207	1.28410	0.74992
$r = 0.8, d = 5$	0.29972	0.42580	0.75337	0.42486	0.70390	1.76931	0.99780
$r = 0.8, d = 6$	0.29922	0.38669	0.78082	0.42880	0.77379	2.01922	1.10888
$r = 2, d = 4$	0.29970	0.53578	0.77402	0.42328	0.55937	1.44466	0.79002
$r = 2, d = 5$	0.30021	0.42405	0.81194	0.42479	0.70796	1.91473	1.00174
$r = 2, d = 6$	0.30050	0.38871	0.83975	0.42698	0.77309	2.16039	1.09847
Panel (c): $\beta_1 \in (-1, 0), \beta_2 \in (0, 1)$							
$r = 0.8, d = 4$	0.29942	0.53158	0.39353	0.34666	0.56327	0.74029	0.65212
$r = 0.8, d = 5$	0.29993	0.43544	0.39468	0.34786	0.68881	0.90640	0.79888
$r = 0.8, d = 6$	0.30003	0.39344	0.39514	0.34948	0.76257	1.00431	0.88827
$r = 2, d = 4$	0.30088	0.53712	0.41771	0.34455	0.56018	0.77770	0.64149
$r = 2, d = 5$	0.29959	0.43570	0.42138	0.34349	0.68760	0.96712	0.78835
$r = 2, d = 6$	0.29957	0.39110	0.42320	0.34473	0.76595	1.08208	0.88144
Panel (d): $\beta_1 \in (0, 1), \beta_2 \in (-1, 0)$							
$r = 0.8, d = 4$	0.29991	0.54170	0.94341	0.48621	0.55365	1.74155	0.89756
$r = 0.8, d = 5$	0.30044	0.42537	0.98715	0.49148	0.70630	2.32067	1.15542
$r = 0.8, d = 6$	0.30079	0.39497	1.02682	0.49603	0.76155	2.59973	1.25588
$r = 2, d = 4$	0.30080	0.52253	1.00837	0.49099	0.57565	1.92978	0.93963
$r = 2, d = 5$	0.29956	0.43413	1.06041	0.49239	0.69004	2.44263	1.13420
$r = 2, d = 6$	0.30043	0.39440	1.10692	0.49708	0.76174	2.80658	1.26033

Table 4: One-step-ahead forecast performance with random structural break at $\theta \in \{2, \dots, d-2\}$, $\eta(t) = \sqrt{12}(\exp(t + 3 \arctan(s)) - \lfloor \exp(t + 3 \arctan(s)) \rfloor - 1/2)$, and $N_{sim} = 3 \times 10^5$.

	e_{ideal}	$e_{AR(1)}$	e_K	e_{KH}	$e_{ideal}/e_{AR(1)}$	$e_K/e_{AR(1)}$	$e_{KH}/e_{AR(1)}$
Panel (a): $\beta_1, \beta_2 \in (0, 1)$							
$r = 0.8, d = 4$	0.27537	0.30305	0.33968	0.22866	0.90865	1.12087	0.75455
$r = 0.8, d = 5$	0.31588	0.49982	0.39241	0.35066	0.63198	0.78510	0.70157
$r = 0.8, d = 6$	0.28991	0.36228	0.39589	0.32990	0.80024	1.09278	0.91063
$r = 2, d = 4$	0.27537	0.30289	0.37362	0.23235	0.90912	1.23352	0.76710
$r = 2, d = 5$	0.31588	0.49950	0.40473	0.34447	0.63239	0.81027	0.68963
$r = 2, d = 6$	0.28991	0.36230	0.41445	0.32531	0.80019	1.14393	0.89790
Panel (b): $\beta_1, \beta_2 \in (-1, 1)$							
$r = 0.8, d = 4$	0.27537	0.33724	0.52333	0.25020	0.81653	1.55182	0.74190
$r = 0.8, d = 5$	0.31588	0.45315	0.66214	0.41907	0.69707	1.46117	0.92478
$r = 0.8, d = 6$	0.28991	0.35271	0.71703	0.42148	0.82194	2.03290	1.19495
$r = 2, d = 4$	0.27537	0.33766	0.56239	0.25452	0.81551	1.66555	0.75377
$r = 2, d = 5$	0.31588	0.45277	0.70309	0.41578	0.69766	1.55286	0.91829
$r = 2, d = 6$	0.28991	0.35254	0.76555	0.42132	0.82235	2.17153	1.19508
Panel (c): $\beta_1 \in (-1, 0), \beta_2 \in (0, 1)$							
$r = 0.8, d = 4$	0.27537	0.30559	0.34224	0.22801	0.90109	1.11993	0.74613
$r = 0.8, d = 5$	0.31588	0.48764	0.39469	0.35113	0.64777	0.80937	0.72005
$r = 0.8, d = 6$	0.28991	0.35639	0.39934	0.33230	0.81346	1.12052	0.93241
$r = 2, d = 4$	0.27537	0.30552	0.37980	0.23065	0.90129	1.24313	0.75495
$r = 2, d = 5$	0.31588	0.48764	0.40948	0.34395	0.64778	0.83973	0.70535
$r = 2, d = 6$	0.28991	0.35629	0.42379	0.32902	0.81369	1.18945	0.92346
Panel (d): $\beta_1 \in (0, 1), \beta_2 \in (-1, 0)$							
$r = 0.8, d = 4$	0.27537	0.37214	0.62981	0.26768	0.73995	1.69240	0.71930
$r = 0.8, d = 5$	0.31588	0.42228	0.83730	0.47762	0.74804	1.98282	1.13106
$r = 0.8, d = 6$	0.28991	0.35600	0.90941	0.49547	0.81435	2.55451	1.39176
$r = 2, d = 4$	0.27537	0.37196	0.67132	0.27473	0.74032	1.80483	0.73860
$r = 2, d = 5$	0.31588	0.42222	0.89167	0.47730	0.74815	2.11187	1.13045
$r = 2, d = 6$	0.28991	0.35638	0.96375	0.49639	0.81349	2.70429	1.39286

Table 5: One-step-ahead forecast performance with random structural break at $\theta \in \{2, \dots, d-2\}$, and $\theta \in \{2, \dots, d-2\}$, $\eta(t) = 2^{-1/2}\eta_0(t) + 2^{-1/2}\eta_0(t-1)$, where $\eta_0(t) \sim \mathcal{N}(0, 1)$, and $N_{sim} = 3 \times 10^5$.

	e_{ideal}	$e_{AR(1)}$	e_K	e_{KH}	$e_{ideal}/e_{AR(1)}$	$e_K/e_{AR(1)}$	$e_{KH}/e_{AR(1)}$
Panel (a): $\beta_1, \beta_2 \in (0, 1)$							
$r = 0.8, d = 4$	0.30035	0.41511	0.26556	0.33784	0.72355	0.63973	0.81386
$r = 0.8, d = 5$	0.30030	0.35935	0.26788	0.33906	0.83566	0.74544	0.94354
$r = 0.8, d = 6$	0.29979	0.33688	0.26966	0.33870	0.88989	0.80046	1.00540
$r = 2, d = 4$	0.30093	0.41192	0.26972	0.33210	0.73057	0.65479	0.80623
$r = 2, d = 5$	0.29996	0.36100	0.27102	0.33213	0.83093	0.75077	0.92005
$r = 2, d = 6$	0.30007	0.33697	0.27332	0.33336	0.89049	0.81111	0.98927
Panel (b): $\beta_1, \beta_2 \in (-1, 1)$							
$r = 0.8, d = 4$	0.29964	0.38414	0.36390	0.32763	0.78002	0.94730	0.85290
$r = 0.8, d = 5$	0.29966	0.33359	0.37153	0.32864	0.89830	1.11373	0.98518
$r = 0.8, d = 6$	0.30046	0.31771	0.37667	0.33065	0.94571	1.18558	1.04076
$r = 2, d = 4$	0.30027	0.38045	0.38446	0.32176	0.78925	1.01055	0.84574
$r = 2, d = 5$	0.29993	0.33348	0.39512	0.32317	0.89940	1.18484	0.96909
$r = 2, d = 6$	0.30055	0.31620	0.39959	0.32357	0.95048	1.26372	1.02329
Panel (c): $\beta_1 \in (-1, 0), \beta_2 \in (0, 1)$							
$r = 0.8, d = 4$	0.30001	0.41772	0.26456	0.33607	0.71820	0.63335	0.80454
$r = 0.8, d = 5$	0.29908	0.37058	0.26511	0.33689	0.80706	0.71538	0.90910
$r = 0.8, d = 6$	0.30038	0.35256	0.26684	0.33917	0.85199	0.75686	0.96201
$r = 2, d = 4$	0.30012	0.41655	0.27130	0.33070	0.72051	0.65132	0.79392
$r = 2, d = 5$	0.30027	0.37022	0.27162	0.33141	0.81107	0.73367	0.89518
$r = 2, d = 6$	0.30058	0.35262	0.27223	0.33241	0.85242	0.77202	0.94269
Panel (d): $\beta_1 \in (0, 1), \beta_2 \in (-1, 0)$							
$r = 0.8, d = 4$	0.29979	0.34089	0.48870	0.32159	0.87942	1.43359	0.94338
$r = 0.8, d = 5$	0.29994	0.30282	0.52276	0.32426	0.99050	1.72634	1.07081
$r = 0.8, d = 6$	0.30005	0.29188	0.53985	0.32555	1.02798	1.84959	1.11535
$r = 2, d = 4$	0.29978	0.34135	0.52265	0.31501	0.87822	1.53112	0.92284
$r = 2, d = 5$	0.30037	0.30242	0.56744	0.31838	0.99322	1.87633	1.05277
$r = 2, d = 6$	0.29971	0.29286	0.59408	0.32005	1.02337	2.02852	1.09284

4 Conclusions

This paper addresses the problem of one-step-ahead forecast of an AR(1) process with a single structural break at an unknown time and of unknown sign and magnitude within a very short learning sequence. We analysed, via simulation experiments, the forecast performance of a smoothed predicting kernel algorithm relative to that of a linear predictor that utilize the AR(1) model parameter estimated from the learning sequence without taking into account the presence of structural break.

It appears that the shorter the learning sequence, the better the forecast performance of the smoothed predicting kernel relative to the linear predictor. Regardless whether the innovation terms in the learning sequences are constructed from IID random Gaussian variables, IID random Gamma variables, IID scaled pseudo-uniform variables, or first-order auto-correlated Gaussian process, the forecast performance of the smoothed predicting kernel is better than that of the linear predictor if the AR(1) model parameter switches from a negative value to a positive value in the learning sequence, i.e., $\beta_1 \in (-1, 0), \beta_2 \in (0, 1)$. However, it is not so for the other regime switching scenarios considered in the simulation experiments, i.e., $\beta_1, \beta_2 \in (0, 1), \beta_1, \beta_2 \in (-1, 1)$, and $\beta_1 \in (0, 1), \beta_2 \in (0, 1)$.

It could be interesting to explore the forecast performance of the smoothed linear predictor in the context of random-coefficient AR(1) process (see, among others, Leipus et al., 2006) where the AR(1) model parameter between any two sequential observations are independent and identically distributed random variables from the uniform distribution $U[0, 1]$. Additionally, we may explore the implementing the smoothed predicting linear predictor in the context of adaptive linear filtering to perform successive, on-line one-step ahead forecast. We leave this for future work.

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